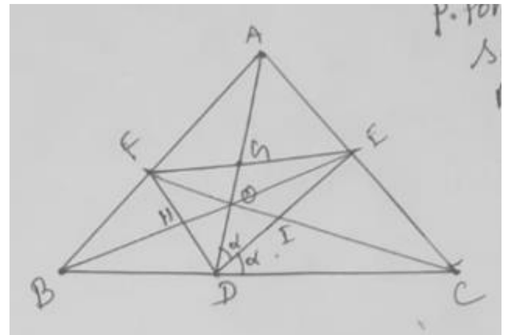
01.02.2024 – III Prize Winner – Mrs.Ponmalar Selvi's Solution



AD & BE are two cevians concurrent at O.

DE is the bisector of $\angle ADC$.

Let $\angle ADE = \angle EDC = \alpha$

Draw CF passing through O which intersect AB at F.

In ΔABC,

AD & FE intersect at G. BE and FD intersect at H and CF and DE intersect at I.

By Unity Pieces Theorem,

$\frac{OD}{AD} + \frac{OE}{BE} + \frac{OF}{CF} = 1$ ------------------------------ (1)

In ΔODC,

OI is the bisector of $\angle ODC$

By ABT, $\frac{DO}{DC} = \frac{OI}{IC}$ ---------------------------------(2)

By Concurrency Theorem,

$\frac{OE}{BE} = \frac{OH}{HB}$ & $\frac{OI}{IC} + \frac{OF}{CF}$ ----------------------------(3)

From (2) & (3)

$\frac{OD}{DC} = \frac{OF}{CF}$ -- (4)

Sub (4) in (1)

$\frac{OD}{AD} + \frac{OE}{BE} + \frac{OD}{DC} = 1$ -----------------------------------(5)

It is enough to prove $\frac{OE}{BE} = \frac{OD}{DB}$

DE is the external bisector ∠ODB.

$$\frac{OD}{DB} = \frac{OE}{EB}$$

(5) $\Rightarrow \frac{DO}{AD} + \frac{OD}{DB} + \frac{OD}{DC} = 1$

OD $\left(\frac{1}{AD} + \frac{1}{BD} + \frac{1}{CD}\right) = 1$

$\frac{1}{AD} + \frac{1}{BD} + \frac{1}{CD} = \frac{1}{OD}$ ------------------ Hence Proved

As DE is the angular bisector,

$\frac{\mathrm{CD}}{\mathrm{AD}} = \frac{\mathrm{CE}}{\mathrm{AE}}$ ----------------------------------(1)

$\Longrightarrow$ For BE & ΔADC,

$$\left|\frac{OA}{OD} \cdot \frac{DB}{BC} \cdot \frac{CE}{AE}\right| = 1$$

$\left[\frac{AD-OD}{OD}\right] \cdot \left[\frac{BD}{BD+CD}\right] \cdot \frac{CD}{AD} = 1$ [from (1)

$$\left[\frac{CD}{OD} - \frac{CD}{AD}\right] \cdot \left[\frac{BD}{BD + CD}\right] = 1$$

$$\frac{CD}{OD} - \frac{CD}{AD} = 1 + \frac{CD}{BD}$$

$$\frac{1}{OD} - \frac{1}{AD} = \frac{1}{CD} + \frac{1}{BD}$$

$\Longrightarrow \frac{1}{AD} + \frac{1}{BD} + \frac{1}{CD} = \frac{1}{OD}$ ----------------------- Hence Proved

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